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PARTIAL ORDERS OF DIMENSION 2, INTERVAL ORDERS, AND INTERVAL GRAPHS

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PARTIAL ORDERS OF DIMENSION 2
INTERVAL ORDERS, AND INTERVAL GRAPHS

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SUMMARY

The dimension $D(<)$ of a partial order $(X,<)$ is defined as the smallest cardinal number m so that $<$ is the intersection of m linear orders. Some new characterizations of partial orders with $D \leq 2$ are obtained, and known characterizations are summarized.

In particular, using the notion of comparability graph defined by Ghouilà-Houri and Gilmore and Hoffman, it is shown that the dimension of a partial order is at most 2 if and only if its incomparability graph is a comparability graph. Equivalently, if $x \sim y$ holds whenever x and y are incomparable under $<$, then $D(<) \leq 2$ if and only if every odd $\sim$-cycle has a triangular chord.

Partial orders with $D \leq 2$ are also related to lattices with planar Hasse diagrams, and a procedure is described for obtaining from a given partial order $(X,<)$ a lattice $L(X,<)$ such that $D(<) \leq 2$ if and only if $L(X,<)$ has a planar Hasse diagram. It is also shown that every finite partial order with $D \leq 2$ has at least one doubly irreducible element.

It is observed that no appreciably simpler axiomatization for the class of partial orders with $D \leq 2$ than those given can be obtained, in particular that this class is not finitely axiomatizable.

Partial orders with $D \leq 2$ are related to other types of binary relations which have been studied in the literature,

including weak orders, interval orders, semiorders, and interval graphs. The latter part of the paper analyzes connections between two-dimensional partial orders and these other relations. It is shown that although interval orders and semiorders can have dimension bigger than 2, $D(<) \leq 2$ when $<$ satisfies the so-called weak interval condition or when $<$ is a so-called strong interval order.

Finally, the notion of dimension of a partial order is related to the notion of breadth B, and it is noted that $B(<) \leq D(<)$ and $B(<) \leq 2$ whenever $<$ is an interval order.

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1. INTRODUCTION

It has long been known that if $<$ is a binary relation on a countable set X then $<$ is a weak order (Section 6) if and only if the following condition holds: there is a real-valued function f on X such that, for all $x, y \in X$,

$$x < y \text{ if and only if } f(x) < f(y).$$

Proofs of this and its extension to X of arbitrary cardinality are given, for example, by Milgram [1939], Birkhoff [1967], Luce and Suppes [1965], and Fishburn [1970a]. An obvious generalization of this arises when we inquire into conditions on $<$ that are necessary and sufficient for the existence of two real-valued functions f and g on X such that, for all $x, y \in X$,

$$x < y \text{ if and only if } f(x) < f(y) \text{ and } g(x) < g(y). \quad (1)$$

It turns out that (1) can hold when X is countable if and only if $<$ is a partial order of dimension ≤ 2. This will follow from our definitions and a later theorem (Theorem 6.1).

Characterizing the representation (1) is a typical problem in the "theory of measurement" and it is interesting from the point of view of measurement of preference and of generalizations of one-dimensional utility functions,

if $\prec$ is thought of as an individual's (or group's) relation of preference. For background on preference and utility, the reader is referred to Fishburn [1970a], and for background on measurement of preference and the theory of measurement in general, good references are Scott and Suppes [1958] and Suppes and Zinnes [1963].

The numerical representation (1) provided the motivation for the research leading to this paper; the connection of (1) with the dimensionality of a partial order then led to further inquiries involving dimensionality. As a consequence, this paper has two main purposes. The first is to illustrate the relationship among dimensionality and several other concepts of interest in the theory of binary relations, including weak orders, interval orders, semiorders, interval graphs, comparability graphs, and the breadth of a partial order. The second purpose is to prove several new results and to indicate topics for further research.

Section 2 defines the dimension of a partial order in terms of the intersection of linear orders and cites several interesting theorems about dimensionality. Section 3 then shows how the early work of Dushnik and Miller [1941] on dimensionality combines with a more recent result to provide a new characterization (axiomatization) of partial orders with dimension not exceeding two, i.e., $D(<) \leq 2$. The ensuing section relates partial orders

with dimension at most 2 and lattices with planar Hasse diagrams. Then Section 5 shows that the class of partial orders with $D \leq 2$ is not finitely axiomatizable.

The main purpose of Section 6 is to prove that, although a weak order may be two-dimensional, the intersection of any number of weak orders exceeding one is equal to the intersection of a comparable number of linear orders. After noting that two special kinds of partial orders (interval orders, semiorders) can have $D > 2$, Section 7 shows that $D(<) \leq 2$ when $<$ satisfies the so-called weak interval condition or when $<$ is a so-called strong interval order.

Section 8 then illustrates several relationships among the notion of an interval graph and the previously defined concepts. Section 9 summarizes the tests for two-dimensionality. The final section examines briefly the notion of the breadth $B(<)$ of a partial order $<$, noting that $B(<) \leq D(<)$ and that $B(<) \leq 2$ whenever $<$ is an interval order.

2. DIMENSION OF A PARTIAL ORDER

With a few obvious exceptions, binary relations are here defined on a given set X. A binary relation $<$ is a partial order if it is irreflexive ($x \nless x$) and transitive ($x < y$ and $y < z$ imply $x < z$). $<$ is a linear order if it is a partial order which is complete ($x \neq y$ implies $x < y$ or $y < x$).

Finally $x \simeq y$ means that $x \nless y$ and $y \nless x$. $\simeq$ is variously referred to as indifference, matching, similarity, and so forth. We shall call it similarity. $x \sim y$ means that $x \neq y$ and $x \simeq y$. This relation is called incomparability. Intersections of binary relations are defined as usual: $(x,y) \in \cap_A <_a$ if and only if $(x,y) \in <_a$ [i.e., $x <_a y$] for all $a \in A$. Clearly, the intersection of a set of linear orders is a partial order. Conversely, every partial order can be realized as the intersection of a set of linear orders, for by Szpilrajn's extension theorem [1930], if $x \sim y$ there are linear orders $<_1$ and $<_2$ that include $<$ and have $x <_1 y$ and $y <_2 x$. Following Dushnik and Miller [1941], the dimension $D(<)$ of a partial order is the smallest cardinal number m such that $<$ equals the intersection of m linear orders.*

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*Ore [1962] uses the term "order dimension" for the Dushnik-Miller notion, and the term "product dimension" for the following equivalent notion due to Hiraguchi [1955]. The product dimension of a partial order $<$ is the least cardinal m such that $<$ can be embedded, as a partial order, in the cardinal product of m chains. It should be men-

With $|X|$ the cardinality of X, $D(<) \leq |X \times X|$, so that $D(<) \leq |X|$ when X is infinite. For finite X, Hiraguchi [1955] shows that $D(<) \leq \lceil |X|/2 \rceil$ when $|X| \geq 3$. Dushnik and Miller [1941] prove that for every cardinal number m there is a partial order (on an appropriate X) with dimension m. Komm [1948] notes that $D(\subset) = |S|$ when X is the set of all subsets of a set S. Ducamp [1967] presents a procedure for determining the dimension of a given partial order that is quite usable when $|X|$ is not too large.

Figure 1 shows three partial orders for $|X| = 6$. In these <u>Hasse diagrams</u> $x < y$ if and only if x lies below y and there is a connected path from x up to y, each of whose links goes upward. By Hiraguchi's result, $D(<_i) \leq 3$ for $i = 1, 2, 3$. Actually, $D(<_1) = D(<_2) = 3$ and $D(<_3) = 2$. $<_1$ is isomorphic to proper inclusion $\subset$ on the set of all subsets of a three-element set minus the empty set and the entire set. From this and Komm's Theorem it is apparent that $D(<_1) = 3$. Two linear orders that verify $D(<_3) = 2$ are $b < c < e < d < f < a$ and $e < f < b < a < c < d$. We shall verify $D(<_2) = 3$ in Section 3, and there discuss more direct ways of ascertaining the dimension of a partial order.

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tioned that these two equivalent notions of dimension of a partial order bear no relation to the "height" or "length" of a partial order, which is also called dimension in some geometrical contexts.

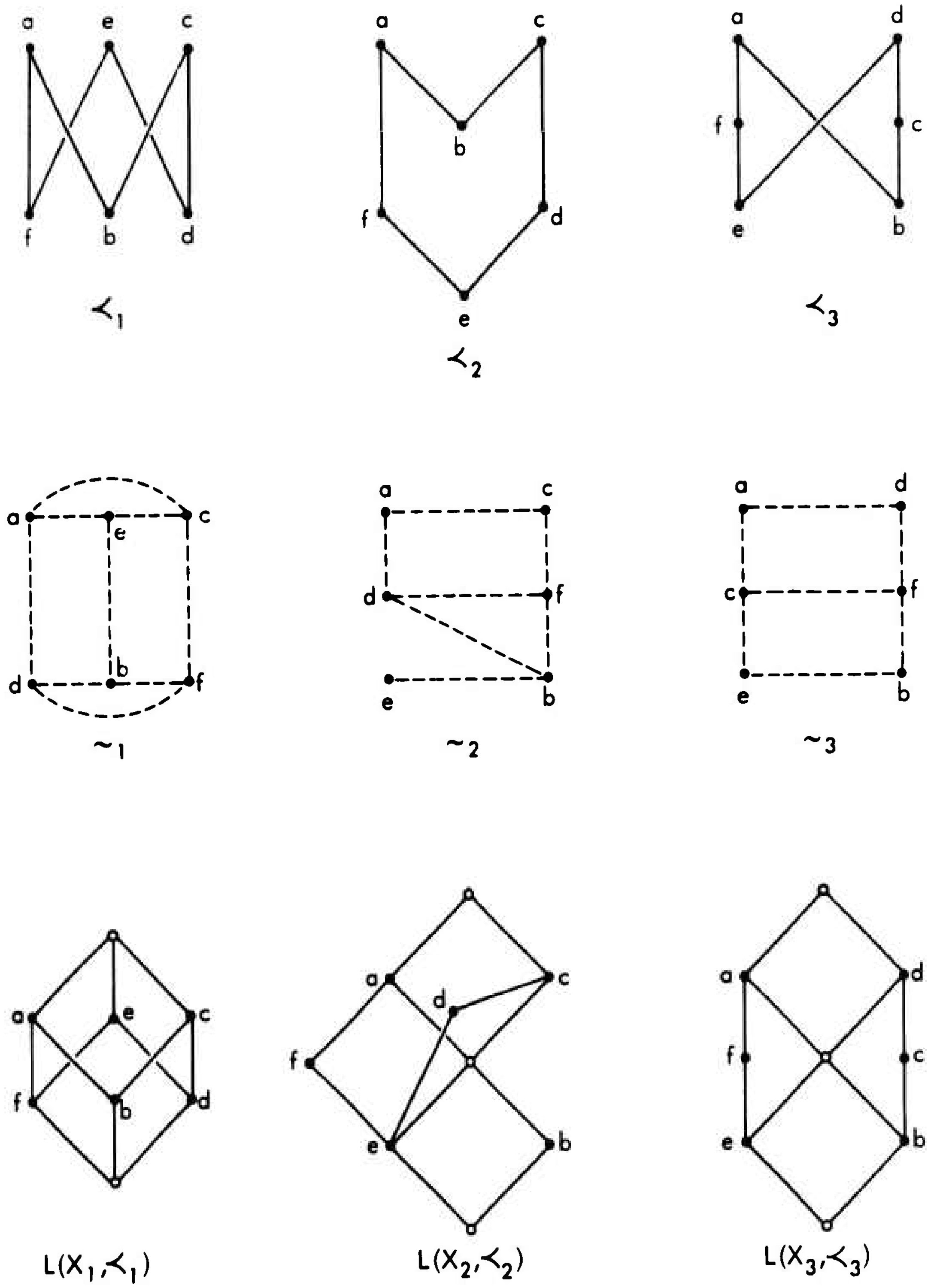
$\prec_1$
$\prec_2$
$\prec_3$
$\sim_1$
$\sim_2$
$\sim_3$
$L(X_1, \prec_1)$
$L(X_2, \prec_2)$
$L(X_3, \prec_3)$

Figure 1.

3. TWO-DIMENSIONAL PARTIAL ORDERS

A characterization of partial orders with $D \leq 2$ has been obtained by Dushnik and Miller [1941]. In this section we combine one of their theorems with a result of Ghouilà-Houri [1962] and Gilmore and Hoffman [1964], hereafter referred to as the G-H Theorem, to obtain a new characterization of the partial orders with $D \leq 2$. It is interesting to note that the results of this section hold without restriction on the cardinality of the underlying set X.

The G-H theorem characterizes the so-called comparability graphs. If $(X,\sim)$ is a <u>graph</u>, i.e. if $\sim$ is a symmetric, irreflexive binary relation on the set X, we say it is a <u>comparability graph</u> if there is a partial order $<^*$ on X such that $x \sim y$ if and only if $x <^* y$ or $y <^* x$. In order to state the G-H theorem, we need the notion of a $\sim$-cycle. We say that $x_0, x_1, \ldots, x_{n-1}$ is a <u>$\sim$-cycle</u> if and only if $x_0 \sim x_1 \sim x_2 \sim \cdots \sim x_{n-1} \sim x_0$, and $x_i = x_j$ for $i \neq j$ implies $x_{i+1} \neq x_{j+1}$, where addition is taken modulo n.* A <u>triangular</u> <u>chord</u> of such a cycle is a pair (x_i, x_{i+2}), where $x_i \sim x_{i+2}$ and where addition is again taken modulo n. The <u>G-H Theorem</u> says that $(X,\sim)$ is a comparability graph if and only if every $\sim$-cycle with an odd number (n) of terms has at least one triangular chord.

To relate this result to partial orders with $D \leq 2$, we note that by a theorem of Dushnik and Miller [1941], $D(<) \leq 2$ if and only if there is a conjugate partial order $<^*$

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*It should be noted that this definition of cycle is slightly more general than the one usually given.

satisfying the following condition: $x <^* y$ or $y <^* x$ if and only $x \sim y$, where $\sim$ is the incomparability relation corresponding to $<$. It follows that a <u>partial order has</u> $D \leq 2$ <u>if and only if its incomparability graph is a comparability graph</u>. Summarizing, we have:

<u>THEOREM 3.1</u>. <u>Suppose</u> $(X,<)$ <u>is a partial order and</u> $\sim$ <u>is its incomparability relation</u>. <u>Then the following are equivalent</u>:

(a). $D(<) \leq 2$.

(b). $(X,\sim)$ <u>is a comparability grapn</u>.

(c). <u>Every odd $\sim$ -cycle has at least one triangular chord</u>.

To verify that $D(<_2) > 2$ for $<_2$ of Fig. 1 we observe that the $\sim_2$-cycle a, c, f, b, e, b, d has no triangular chord since $a \not<_2 f$, $c \not<_2 b$, $f \not<_2 e$, $b = b$, $e \not<_2 d$, $b \not<_2 a$, and $d \not<_2 c$.

To get a better picture of those partial orders which violate the conditions of Theorem 3.1, let us define a <u>comparability cycle</u> in a partial order $(X,<)$ to be a sequence $x_0, x_1, \ldots, x_{2k} \in X$ of odd length $n = 2k+1$ satisfying these three conditions for all i:

(i) x_i and x_{i+1} are comparable, i.e., $x_i < x_{i+1}$ or $x_{i+1} < x_i$ or $x_i = x_{i+1}$;

(ii) x_i and x_{i+k} are incomparable;

(iii) if $x_i = x_j$ for some $j \neq i$, then $x_{i+k} \neq x_{j+k}$.

Here subscripts are interpreted modulo n. For each i, x_{i+k} is one of the two elements which are "opposite" x_i in the cycle. Condition (ii) thus guarantees that pairs of opposite elements are incomparable. Condition (iii) ensures that the cycle cannot be trivially "short circuited."

COROLLARY 3.2. *Let* $(X, <)$ *be a partial order. Then the following condition is equivalent to conditions* (a), (b), (c) *of Theorem* 3.1:

(d) $(X, <)$ *has no comparability cycle*.

Proof. If $x_0, x_1, \ldots, x_{2k}$ is a comparability cycle and $n = 2k + 1$, then $x_0, x_k, x_{2k}, x_{3k}, \ldots, x_{(n-1)k}$ is an odd $\sim$-cycle with no triangular chords. (Here subscripts are computed modulo n.) Conversely, if $y_0, \ldots, y_{n-1}$ is an odd $\sim$-cycle with no triangular chords then $y_0, y_2, y_4, \ldots, y_{n-1}, y_1, y_3, \ldots, y_{n-2}$ is a comparability cycle.

As an application, consider again the partial orders $<_1$, $<_2$ of Fig. 1. In both cases, the elements in alphabetical order make a comparability cycle if one element is repeated; e.g., a, a, b, c, d, e, f. Therefore both $<_1$ and $<_2$ have dimension greater than 2.

In view of the fact that $D(<) \leq 2$ is not finitely axiomatizable (Section 5), we doubt that a significantly simpler set of necessary and sufficient conditions for

$D(\prec) \leq 2$ than those of this section can be given. We do not presently know of theorems like Theorem 3.1 for dimensionality greater than 2.

4. LATTICES WITH PLANAR HASSE DIAGRAMS

We recall that a lattice is a partial order $(X, <)$ such that every two elements $x, y \in X$ have a least upper bound in X, denoted $x \vee y$, and a greatest lower bound, denoted $x \wedge y$. The element $x \vee y$ is usually called the join of x and y, $x \wedge y$ the meet of x and y.

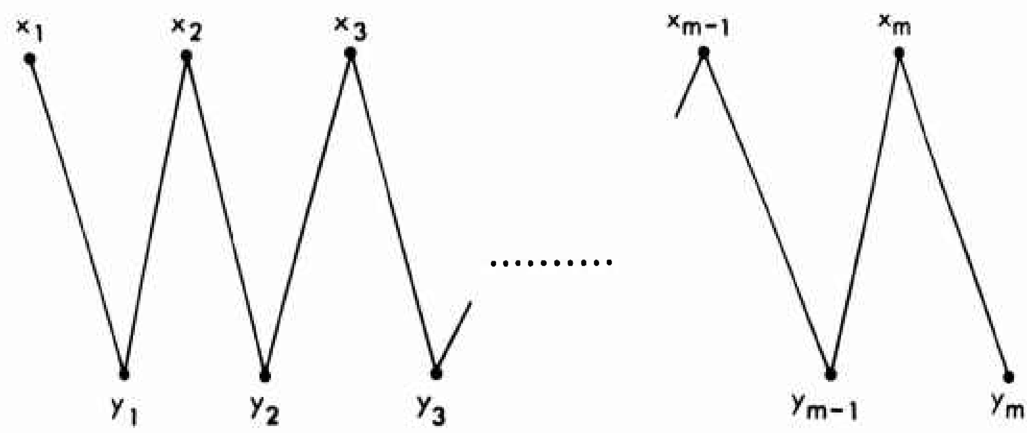
Let us say that a lattice $(X,<)$ has a planar Hasse diagram if a Hasse diagram for $<$ can be drawn in the real plane so that $a < b$ implies that a has a smaller y-coordinate than b and so that no "covering lines" cross. As the reader can readily verify, the lattice $L(X_3,<_3)$ of Fig. 1 has a planar Hasse diagram and the lattices $L(X_1,<_1)$ and $L(X_2,<_2)$ do not. It has been observed by J. Zilber (Birkhoff [1967, p. 32, ex. 7c]) that a finite lattice $(X,<)$ has a planar Hasse diagram if and only if there is a conjugate partial order $<*$ in the sense of Dushnik and Miller (cf. Section 3). (Intuitively, the conjugate partial order goes from left to right in the plane, rather than from bottom to top). Thus a finite lattice has dimension at most 2 if and only if it has a planar diagram.

The planar diagram test for 2-dimensionality does not hold for finite partial orders in general. The partial order $<_2$ of Fig. 1 has a planar diagram but its dimension is 3. The planar diagram test will be useful if we can find a method of constructing a finite lattice having the same dimension as a given partially ordered set. Suppose

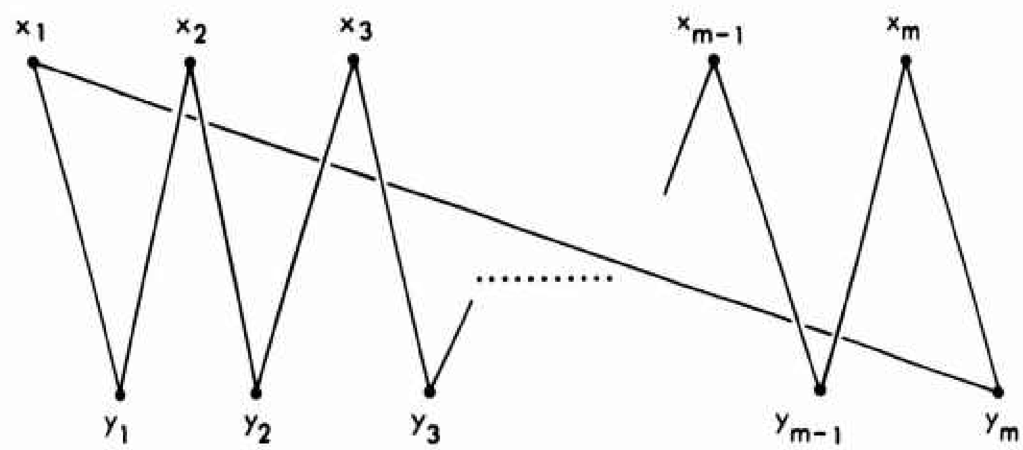
a partial order $(X,<)$ has the property that each pair of elements $x, y \in X$ either has a least upper bound or no common upper bound at all, and similarly for lower bounds. Then a lattice is obtained by simply adjoining a top and a bottom element to X. For example, this is true of the "crown and fence" partial orders of Fig. 2.

For finite partially ordered sets in general, MacNeille's process of "completion by cuts" (Birkhoff [1967]) constitutes a suitable, simple method, of which the construction in the preceding paragraph is a special case, for obtaining a lattice of the same dimension as a given partial order. (See Baker [1961]). It is interesting to present this method more generally, in terms of directed graphs. First, we define the dimension of such a graph G. Let G^* be the set of strongly connected components of G. The existence of a path from one strong component to another defines a natural partial order $<^*$ on G^*. Let us define $D(G) = D(<^*)$. Thus, if G is the Hasse diagram of a finite partial order $(X,<)$ (with G regarded as a directed graph), or if G is the full graph of the order relation on such a set, then $D(G) = D(<)$.

We now construct a lattice L(G) of the same dimension as G. (In this context, L(G) might appropriately be called the "accessibility lattice" of G.) L(G) is the set of closed subsets of G, ordered by inclusion, where "closed" will be defined below. In G, we write $x \longrightarrow y$ if $x = y$ or



$\prec^m$, the <u>fence</u> <u>o</u>˙ <u>order</u> m, $m \geqslant 3$.



$\prec_m$, the <u>crown</u> <u>of</u> <u>order</u> m, $m \geqslant 3$.

Figure 2.

if there is a directed path from x to y. For any subset S of V(G), the vertices of G, let S' denote $\{x \in V(G)$: if $s \longrightarrow y$ for all $s \in S$, then $x \longrightarrow y\}$. In other words, S' is the largest subset of V(G) such that y is accessible from every point of S' if and only if y is accessible from every point of S. Clearly, $S \subseteq S'$, $S'' = S'$, and $S \subseteq T$ implies $S' \subseteq T'$. This suggests calling S <u>closed</u> if $S' = S$. The closed sets are then those of the form S' for some S. Since any intersection of closed sets is closed, the closed sets form a lattice L(G) which is easily calculated. More formally, L(G) is the lattice of sets closed under the Galois connection generated by the relation $x \longrightarrow y$ (cf. Ore [1962, Ch. 11]). If $(X,<)$ is a finite partial order with Hasse diagram G, let us write $L(X,<)$ for L(G). By Baker [1961], $D(<) = D\left(L(X,<)\right)$. To summarize:

<u>THEOREM 4.1</u>. <u>Let</u> $(X,<)$ <u>be a finite partial order</u>. <u>Then</u> $D(<) \leq 2$ <u>if and only if</u> $L(X,<)$ <u>has a planar Hasse diagram</u>.

In this context, it is easy to derive a useful property of partial orders of dimension ≤ 2. In a finite partial order $(X,<)$, we say that x is <u>join irreducible</u> if x is not the least upper bound of two smaller elements, x is <u>meet-irreducible</u> if x is not the greatest lower bound of two larger elements, and x is <u>doubly irreducible</u> if x is both join- and meet- irreducible. In the Hasse diagram of

a finite lattice, such elements are easy to identify visually:

LEMMA 4.2. *Let $(L,<)$ be a finite lattice. Then the following are equivalent for $x \in L$.*

(a) *x is doubly irreducible in L.*

(b) *x covers at most one element of L and is covered by at most one element of L.*

(c) *In the Hasse diagram of L, at most one edge is drawn to x from below and at most one edge is drawn to x from above.*

(d) *$L - \{x\}$ (i.e., L with x deleted) is a sublattice of L or is empty.*

The reader is warned that in a finite partial order which is not a lattice, doubly irreducible elements are not so easy to identify. For example, the element b of $<_2$ in Fig. 1 is doubly irreducible, although at first glance it appears to be a meet of a and c. In fact, a and c have no meet, since both b and e are common lower bounds and are incomparable.

THEOREM 4.3. *Let $(X,<)$ be a finite partial order with $D(<) \leq 2$. Then X has at least one doubly irreducible element.*

Proof. Consider first the case where $(X,<)$ is a lattice. Intuitively, we shall draw X in the plane and then

follow the left side of X up until we come to the desired element. Accordingly, let $<^*$ be a conjugate partial order for X, and let $S = \{x \in X\colon x$ is minimal with respect to $<^*\}$. Then S is a maximal totally unordered set in $(X,<^*)$, hence a maximal chain in $(X,<)$. Let $s_0, \ldots, s_n$ be the elements of S in increasing $<$-order.

An observation: For any i, either s_i is meet-irreducible or s_{i+1} is join-irreducible. Otherwise, since S is a maximal $<$-chain, we would have $s_i = s_{i+1} \wedge a$, $s_{i+1} = s_i \vee b$, with $\{s_i, b\}$, $\{s_{i+1}, a\}$ each being a $<$-incomparable pair. Moreover, $\{a, b\}$ is a $<$-incomparable pair. For if $b < a$, then since $s_{i+1} = s_i \vee b$, we have $s_{i+1} < a$, whence s_{i+1}, a are not $<$-incomparable. And if $a < b$, then $s_i < b$, whence s_i, b are not $<$-incomparable. By the definition of S, $s_i <^* b$ and $s_{i+1} <^* a$, and so we have either $s_i <^* b <^* a$ or $s_{i+1} <^* a <^* b$ in contradiction of $s_i < a$ or of $b < s_{i+1}$.

Now by the maximality of the chain S, s_0 is a minimal element of $(X,<)$ and so is join-irreducible. Thus there is a greatest i for which s_i is join-irreducible. If $i < n$, then s_i is also meet-irreducible, by the observation. If $i = n$, s_i is meet-reducible because by the maximality of the chain S, s_n is maximal in $(X,<)$.

For the case of a general finite partial order $(X,<)$, the above argument shows that at least $L(X,<)$ has a doubly irreducible element. But by the construction of $L(X,<)$, every element of $L(X,<)$ not in X is either a join or a meet of elements in X.

COROLLARY 4.4. *If* $(X,<)$ *is a finite partial order and* X *or some subset of* X *under the restricted partial order has no doubly irreducible element, then* $D(<) > 2$.

As an application, consider the "crown" partial order $<_m$ of Fig. 2. Every element can be expressed as a meet or as a join of other elements; hence $D(<_m) > 2$. However for the "degenerate crown" obtained by allowing m to be 2, all four elements are acutally doubly irreducible.

COROLARY 4.5. *If* $L = (X,<)$ *is a finite lattice with* $D(<) \leq 2$, *then* L *can be "dismantled" by removing one element at a time, each time with a sublattice of* L *remaining*.

This corollary does not state a sufficient condition for $D(X,<) \leq 2$, however. For example, the lattice $L(X_2,<_2)$ of Fig. 1 has dimension 3 but can be dismantled in the same sense.

5. COMMENTS ON AXIOMS FOR $D \leq 2$.

The characterization of partial orders with $D \leq 2$ presented in Section 3 is not a finite characterization in the following sense. It is not possible to decide, using this characterization, whether a given partial order has $D \leq 2$ simply by checking a finite list of forbidden partial orders to see whether one of these partial orders is contained in (more specifically, is isomorphic to a restriction of) the given one. It turns out that no characterization finite in this sense exists, and this we set out to prove in the present section.

The problem of characterizing partial orders with $D \leq 2$ turns out to be equivalent, when X is countable, to characterizing partial orders satisfying the representation (1). Characterizing such partial orders is, as was pointed out earlier, a typical problem in the "theory of measurement." Scott and Suppes [1958], in their foundational paper on this subject, discuss why it is important to obtain characterizations for problems such as this which are finite in the sense described above. We refer the reader to that discussion.

We shall now state and prove two closely related theorems on the axiomatization of $D \leq 2$.

THEOREM 5.1. *There is no finite list of partial orders* $(X_i, <_i)$ *such that for any partial order* $(X, <)$, $D(<) \leq 2$

<u>if and only if</u> $(X,<)$ <u>contains no restriction isomorphic to one of the</u> $(X_i,<_i)$.

In fact,

THEOREM 5.2. <u>The collection of all partial orders</u> $(X,<)$ <u>for which</u> $D(<) \leq 2$ <u>is not axiomatizable by a sentence of first-order logic</u>.

We hasten to add that both these results can immediately be generalized to partial orders with $D \leq n$ for arbitrary $n > 1$ in place of $n = 2$.

<u>Proofs</u>. We shall use the crown and fence of each order m as defined in Fig. 2. Both crowns and fences become lattices if a top and a bottom element are adjoined. Clearly this lattice-completion of a fence has a planar diagram; fences therefore have dimension 2. (In terms of linear orders, take $y_1 < x_1 < y_2 < x_2 < \ldots < y_m < x_m$ and $y_m <' y_{m-1} <' x_m <' y_{m-2} <' x_{m-1} <' y_{m-3} <' \ldots <' y_2 <' x_3 <' y_1 <' x_2 <' x_1$; the intersection of these two linear orders is $<^m$.) On the other hand, as discussed in Section 4, a crown has dimension $D(<_m) > 2$, since a crown has no doubly irreducible elements. (It is easy to check that in fact $D(<_m) = 3$.)

Now, to prove Theorem 5.1, we simply note that for all m, exactly the same partial orders occur as restrictions of $<_m$ and $<^m$ to subsets of at most 2m-1 elements. Thus, if any

finite list of partial orders is given and 2m is larger than the size of all the finite members of the list, then the list cannot be used to distinguish between the crown and the fence of order m. But one of these has dimension 3 and the other, dimension 2.

Theorem 5.2 follows from Theorem 5.1. To see this, note that since the property "$<$ is a partial order of dimension at most 2" is preserved under restriction to subsets, any sentence of first-order logic characterizing this property would be equivalent to a universal sentence, according to a well-known result of Los and Tarski (cf. Grätzer [1968, p. 274]). But a theorem of Vaught [1954] shows that if a collection of partial orders is axiomatizable by a universal sentence, there is an integer n such that, for any partial order $<$, $<$ is in the collection if and only if the restriction of $<$ to every subset of at most n elements is also in the collection. Thus, the set of (nonisomorphic) partial orders on at most n elements and having dimension greater than 2 would constitute a finite list of the kind proscribed by Theorem 5.1.

An alternative, more direct proof of Theorem 5.2 is to observe that an ultraproduct of all crowns is isomorphic to an ultraproduct of all fences, and yet ultraproducts preserve first-order sentences.

To replace 2 by $n \geq 2$ in these theorems, it suffices to note that if there were a finite test for "$D(<) \leq n$",

then "$D(<) \leq 2$" could be tested by checking whether $D(<' \times \underset{\sim}{2}^{n-2}) \leq n$, where $<'$ is the partial order obtained from $<$ by adjoining a top and a bottom element and the expression $<' \times \underset{\sim}{2}^{n-2}$ represents the cardinal product of $<'$ with n-2 two-element chains. (See Baker [1961] for a discussion of the definitions and notation.)

REMARKS.

1. The theorems of this section show, among other things, that the denumerable set {a $\sim$ -cycle with n elements has at least one triangular chord: n = 3, 5, 7, ...} of axioms in Theorem 3.1 cannot be replaced by any finite subset of such axioms. It should be noted that each of these axioms can be written as a universal sentence.

Similar results hold for "$D(<) \leq n$." A suitable countable list of (universal) axioms consists of those expressing the non-containment of the (countably many) finite partial orders of dimension greater than n. Indeed, it can be shown by an ultraproduct construction that the dimension of an arbitrary partial order is the sup of the dimensions of its finite suborders, if all infinite cardinals are lumped together as "∞."

2. Theorems 5.1 and 5.2 hold as well for the case where the (X,<) are assumed to be lattices. A similar "crown-and-fence" proof applies, if top and bottom elements are adjoined to all participants. We have

<u>THEOREM 5.3</u>: <u>There is no finite list of partial orders</u> $(X_i, <_i)$ <u>such that for any lattice</u> $(X,<)$, $D(<) \leq 2$ <u>if and only if</u> $(X,<)$ <u>contains no restriction isomorphic to one of the</u> $(X_i, <_i)$.

<u>THEOREM 5.4</u>. <u>The collection of all lattices</u> $(X,<)$ <u>for which</u> $D(<) \leq 2$ <u>is not axiomatizable by a sentence of first-order logic</u>.

6. WEAK ORDERS

A binary relation $<$ is a <u>weak order</u> if it is asymmetric ($x < y$ implies $y \not< x$) and if $x < y$ implies $x < z$ or $z < y$ (for any $z \in X$). The latter property is referred to by Chipman [1970] as <u>negative transitivity</u> since it is equivalent to $x \not< y$ and $y \not< z$ imply $x \not< z$. It is easy to see that every weak order is a partial order. It is well known that if $<$ is a weak order then $\simeq$ is an equivalence and $x < z$ if either ($x \simeq y$ and $y < z$) or ($x < y$ and $y \simeq z$). Consequently, $<'$ defined on the set $X/\simeq$ of equivalence classes of X under $\simeq$ by [$a <' b$ if and only if $x < y$ for some $x \in a$ and $y \in b$] is a linear order.

It follows easily from the Dushnik–Miller conjugate theorem that $D(<) \leq 2$ when $<$ is a weak order. An example of a two-dimensional weak order is $X = \{x,y\}$ with $x \sim y$.

Despite the fact that a weak order is not necessarily a linear order, the intersection of any number of weak orders exceeding one is equal to the intersection of a like number of linear orders. This leads directly to the result that, when X is countable, condition (1) of Sec. 1 holds if and only if $D(<) \leq 2$. All orders in the following theorem are on the same set.

<u>THEOREM 6.1</u>. <u>If</u> A <u>is a set of weak orders with</u> $|A| > 1$ <u>then there is a set</u> A^* <u>of linear orders such that</u> $|A^*| \leq |A|$ <u>and</u> $\cap_{A^*} <_a = \cap_A <_a$.

PROOF. Suppose first that $|A| = 2$ with $A = \{<_1, <_2\}$. Define $<_1'$ on X by

$$x <_1' y \text{ if and only if } x <_1 y \text{ or } (x \simeq_1 y \text{ and } y <_2 x).$$

It follows easily that $<_1'$ is a weak order. Similarly $<_2'$ defined by

$$x <_2' y \text{ if and only if } x <_2 y \text{ or } (x \simeq_2 y \text{ and } y <_1 x)$$

is a weak order. Moreover $<_1' \cap <_2' = <_1 \cap <_2$. In addition, $x \simeq_1' y$ if and only if $(x \simeq_1 y$ and $x \simeq_2 y)$, and $x \simeq_2' y$ if and only if $(x \simeq_1 y$ and $x \simeq_2 y)$. Thus $X/\simeq_1' = X/\simeq_2'$. For each $\alpha \in X/\simeq_1'$, let $<(\alpha)$ be a linear order on α and $<^c(\alpha)$ be its converse. Obtain the linear orders $<_1^*$ and $<_2^*$ from $<_1'$, $<_2'$ respectively by ordering points within the equivalence class α according to $<(\alpha)$, $<^c(\alpha)$ respectively. It is easy to verify that $<_1^* \cap <_2^* = <_1' \cap <_2'$ and hence that $<_1^* \cap <_2^* = <_1 \cap <_2$.

We complete the proof in case A is finite by induction on $|A|$. Thus, suppose $A = \{<_1, <_2, \ldots, <_k\}$. We know the result for $k = 2$ and we assume it for sets smaller than A. Then there are linear orders $<_1'$, $<_2'$ so that $<_1 \cap <_2 = <_1' \cap <_2'$. And there are by inductive assumption linear orders $<_2''$, $<_3''$, $\ldots$, $<_k''$ so that $<_2' \cap <_3 \cap \ldots \cap <_k = <_2'' \cap <_3'' \cap \ldots \cap <_k''$. Thus, take $A^* = \{<_1', <_2'', <_3'', \ldots, <_k''\}$.

Suppose next A is infinite. For each weak order $\prec$ on A, and each $\alpha \in X/\sim$, let $\prec(\alpha)$ be an arbitrary linear order on α and $\prec^c(\alpha)$ its converse. Obtain linear orders $<'$ and $<''$ from $<$ by ordering points within the equivalence class α by $<(\alpha)$ and $<^c(\alpha)$, respectively. Then $A^* = \{<',<'': < \in A\}$ has the property that $\cap_{A^*} <_a = \cap_A <_a$. Moreover, $|A^*| = |A|$ because A is infinite.

The next section discusses several partial orders that can be thought of as generalizations of weak orders and discusses dimensionality for these generalizations. Theorem 6.1 will be used in the proof of Theorem 7.3.

7. INTERVAL ORDERS AND SEMIORDERS

An interval order is a partial order $<$ for which $x < y$ and $z < w$ imply $x < w$ or $z < y$. The name for this derives from the fact, as proved in Fishburn [1970b], that if X is countable then $<$ on X is an interval order if and only if it satisfies the interval condition, i.e., there is a function F on X to the set of closed real intervals such that, for all $x, y \in X$,

$$x < y \text{ if and only if } \sup F(x) < \inf F(y). \qquad (2)$$

Extension of this representation for X of arbitrary cardinality is pursued in Fishburn [1969b].

An interval α in a linear order $(X^*,<^*)$ is a nonempty subset of X^* for which $a, c \in \alpha$ and $a <^* b <^* c$ imply $b \in \alpha$. The interval is closed if there are $a, b \in X^*$ so that $\alpha = \{a,b\} \cup \{c \in X^*: a <^* c <^* b\}$. It is not too hard to prove, using the compactness theorem for first order predicate calculus, that a partial order $(X,<)$ is an interval order if and only if there is a linear order $(X^*,<^*)$ and a function F on X to the set of closed intervals in $(X^*,<^*)$ so that (2) holds.

An interval order can be viewed also as a generalization of a semiorder as introduced by Luce [1956]. Using the definition in Scott and Suppes [1958], a semiorder is an irreflexive binary relation $<$ that satisfies (i) $x < y$

and $z \prec w$ imply $x \prec w$ or $z \prec y$, and (ii) $x \prec y$ and $y \prec z$ imply $x \prec w$ or $w \prec z$ (for any $w \in X$). A semiorder is a partial order since transitivity follows from irreflexivity and (i). Irreflexivity and (i) fully characterize an interval order. It follows from the main semiorder representation theorem of Scott and Suppes that if X is finite and if $\prec$ is a semiorder then (2) holds with all intervals having the same length. A different proof is given by Scott [1964]. Additional theory for semiorders is found in Holland [1966a,b], Krantz [1967], Roberts [1968, 1969a, 1969b, 1969c], Domotor [1969], and Fishburn [1969b, 1970b].

The $D \leq 2$ property does not hold for interval orders and semiorders. Figure 3 shows a 7-element semiorder that has the odd $\sim$-cycle c, f, c, g, d, a, d, b, e with no triangular chord. Hence, by Theorem 3.1, this semiorder has $D > 2$. We do not presently know if there is an upper bound on the dimensions of semiorders or interval orders.

The connection between the representation (2) and the partial orders with $D \leq 2$ can be made more precise by recalling a result of Dushnik and Miller [1941], namely:

<u>THEOREM 7.1 (Dushnik and Miller)</u>. <u>If</u> $(X,\prec)$ <u>is a partial order, then</u> $D(\prec) \leq 2$ <u>if and only if</u> $\prec$ <u>is realizable as the partial order of inclusion on a family of intervals in some linear order, i.e., there is a linear order</u> $(X',\prec')$

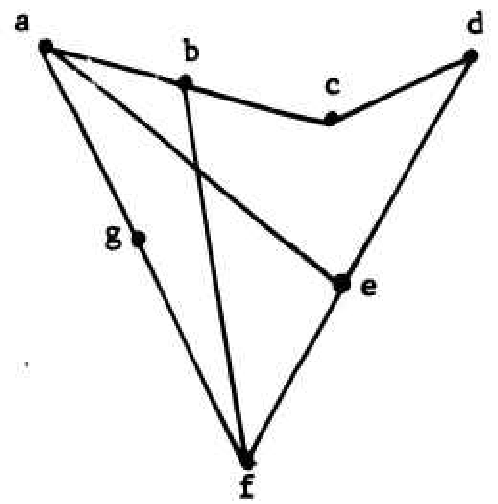


FIGURE 3

<u>and a function</u> F <u>from</u> X <u>into the set of intervals in</u> $(X',<')$ <u>so that for all</u> $x, y \in X$,

$$x < y \text{ if and only if } F(x) \subset F(y). \qquad (3)$$

Here, $\subset$ denotes strict inclusion. It follows from this theorem that if $(X,<)$ has $D \leq 2$, then a conjugate partial order $<*$ is given by

$$x <* y \text{ if and only if}$$
$$[\sup F(x) <' \sup F(y) \ \& \ \inf F(x) <' \inf F(y)]. \qquad (4)$$

We shall say that a partial order $(X,<*)$ satisfies the <u>weak interval condition</u> if it satisfies (4) for some linear order $(X',<')$ and some F mapping X into the set of closed intervals of $(X',<')$.

<u>THEOREM 7.2</u>. <u>A partial order</u> $(X,<)$ <u>satisfies the weak interval condition if and only if</u> $D(<) \leq 2$.

<u>Proof</u>. If $(X,<)$ satisfies the weak interval condition, then the partial order of inclusion on the intervals in the range of F induces a partial order conjugate to $<$, whence $D(<) \leq 2$.

To prove the converse, suppose $D(<) \leq 2$. The proof is similar to Dushnik and Miller's proof of Theorem 7.1. There are linear orders $<_1$ and $<_2$ so that $< = <_1 \cap <_2$.

Let Y be a disjoint copy of X and let $<_2'$ be a copy of $<_2$ on Y. Let $X' = X \cup Y$ and define a linear order $<'$ on X' as follows: $<'$ is $<_1$ on X and $<_2'$ on Y; if $a \in Y$ and $b \in X$, then $a <' b$. Finally, if $\hat{x}$ is the copy of x in Y, define $F(x) = [\hat{x}, x]$, i.e., the closed interval in $(X', <')$ with end points $\hat{x}$ and x. It is not hard to show that $F(x)$ satisfies (4) (with $<$ in place of $<*$).

Despite the negative result that semiorders and interval orders may have dimension greater than 2, there is one relation which is more general than a weak order and less general than an interval order that is easily shown to have $D \leq 2$. For lack of another name we shall call this a <u>strong interval order</u>, by which we mean that $<$ is a partial order for which

$$(x < y \text{ and } z < w \text{ and } x \sim z \text{ and } y \sim w) \text{ imply }$$
$$(x < w \text{ and } z < y).$$

It is readily verified that a strong interval order is an interval order.

<u>THEOREM 7.3</u>. <u>If $<$ is a strong interval order then</u> $D(<) \leq 2$.

<u>Proof</u>. It is shown in Fishburn [1970a, 1970b] that if $<$ is an interval order then $<_1$ and $<_2$ defined by

$x \prec_1 y$ if and only if $x \simeq t$ and $t \prec y$ for some $t \in X$

$x \prec_2 y$ if and only if $x \prec v$ and $v \simeq y$ for some $v \in X$

are weak orders and $x \prec y$ implies $x \prec_1 y$ and $x \prec_2 y$. Conversely, suppose that $x \prec_1 y$ and $x \prec_2 y$. Then by the special property for a strong interval order, $x \prec y$. Hence with $\prec$ a strong interval order, $x \prec y$ if and only if $x \prec_1 y$ and $x \prec_2 y$, or $\prec = \prec_1 \cap \prec_2$. Theorem 6.1 completes the proof.

8. INTERVAL GRAPHS

There is a close relationship among two dimensional partial orders, interval orders, and interval graphs. Interval graphs have been investigated by, among others, Benzer [1959], Lekkerkerker and Boland [1962], Gilmore and Hoffman [1964], Fulkerson and Gross [1965], and Roberts [1969b]. When X is finite and $\sim$ is an irreflexive binary relation on X, $(X,\sim)$ is an <u>interval graph</u> if and only if the following condition holds: there is a function F on X into the set of closed real intervals such that, for all $x \neq y \in X$,

$$x \sim y \text{ if and only if } F(x) \cap F(y) \neq \varphi. \qquad (5)$$

Characterizations of such graphs, in terms of axioms for $\sim$, are given in the aforementioned papers.

A more general definition of an interval graph is given by Gilmore and Hoffman. By their definition, if $\sim$ is irreflexive, $(X,\sim)$ is an <u>interval graph</u> if and only if the following condition holds: there is a linearly ordered set $(X',<')$ and a function G on X into the closed intervals of $(X',<')$ such that for $x \neq y \in X$,

$$x \sim y \text{ if and only if } G(x) \cap G(y) \neq \varphi.$$

This is equivalent to the preceding definition when X is finite.

Using the more general definition we can establish a connection between interval orders and interval graphs. To do this we recall the Gilmore-Hoffman [1964] characterization of interval graphs. If $(X,\sim)$ is a symmetric, irreflexive binary relation, then its irreflexive complement is the relation $(X,\sim')$ where $x \sim' y$ if and only if $x \not\sim y$ and $x \neq y$.

THEOREM 8.1 (Gilmore and Hoffman): *Let $\sim$ be a symmetric, irreflexive binary relation on X and let $\sim'$ be its irreflexive complement. Then $(X,\sim)$ is an interval graph if and only if every odd $\sim'$-cycle has a triangular chord and, whenever x, y, z, and w in X are distinct and satisfy $x \sim y \sim z \sim w \sim x$, then either $x \sim z$ or $y \sim w$.*

THEOREM 8.2. *A partial order $<$ on X is an interval order if and only if its incomparability graph $(X,\sim)$ is an interval graph.*

PROOF. If $X' \subseteq X$, $(X',<)$ and $(X',\sim)$ will denote the restrictions of $<$ and $\sim$ to X'. The theorem is proven in the case that X is finite (countable) in Fishburn [1970b]. From this, from Theorem 8.1, and from the definition of interval order, we see that if $<$ is a partial order then $(X,<)$ is an interval order $\Longleftrightarrow$ for each finite $X' \subseteq X$ $(X',<)$ is an interval order $\Longleftrightarrow$ for each finite $X' \subseteq X$ $(X',\sim)$ is an interval graph $\Longleftrightarrow$ $(X,\sim)$ is an interval graph.

Theorems 8.1 and 3.1 yield several results connecting partial orders with $D \leq 2$ and interval graphs.

THEOREM 8.3. *Suppose $\prec$ is a partial order with $D \leq 2$, $\sim$ is its incomparability relation, $\simeq$ is its similarity relation, and $\sim'$ is the irreflexive complement of $(X,\sim)$. Then $(X,\sim')$ is an interval graph if and only if when x, y, z, and w are all different, $(x \prec z,\ x \prec w,\ y \prec z,\ y \prec w)$ implies either $x \not\simeq y$ or $z \not\simeq w$.*

Proof. If $D(\prec) \leq 2$ then, by Theorem 3.1, every odd $\sim$ -cycle has a triangular chord. Thus, by Theorem 8.1, $(X,\sim')$ is an interval graph if and only if, when, x, z, y and w are all different, $x \sim' z \sim' y \sim' w \sim' x$ implies $x \sim' y$ or $z \sim' w$. This is the same as saying that $[(x \prec z \text{ or } z \prec x) \text{ and } (z \prec y \text{ or } y \prec z) \text{ and } (y \prec w \text{ or } w \prec y) \text{ and } (w \prec x \text{ or } x \prec w)]$ implies $x \prec y$ or $y \prec x$ or $z \prec w$ or $w \prec z$. The only two cases where these hypotheses are consistent (do not violate the partial order axioms) and do not yield $x \prec y$ or $y \prec x$ or $z \prec w$ or $w \prec z$ by the partial order axioms are $(x \prec z,\ y \prec z,\ y \prec w,\ x \prec w)$ and its dual. Hence the $\sim'$ condition can be replaced by the condition in the latter part of the theorem.

THEOREM 8.4. *Suppose $(X,\sim)$ is an interval graph and every odd $\sim$-cycle has a triangular chord. Then there is a partial order $\prec$ for which $\sim$ is the incomparability relation and every such partial order is an interval order with $D \leq 2$.*

<u>Proof</u>. By Theorem 8.1 and the G-H Theorem on comparability graphs mentioned in Sec. 3, the irreflexive complement $(X,\sim')$ of $(X,\sim)$ is a comparability graph and so there is a partial order $<$ such that $\sim$ is its incomparability relation. By Theorem 8.2, any such partial order $<$ is an interval order. Finally, by the G-H Theorem $(X,\sim)$ is a comparability graph and so by Theorem 3.1, $<$ has $D \leq 2$.

It should be mentioned that the hypotheses of Theorem 8.4 do not imply that any partial order $<$ for which $\sim$ is the incomparability relation is a <u>strong</u> interval order. The example of $X = \{x,y,z,w\}$ with $\sim$ defined by $x \sim z$, $y \sim w$, $z \sim y$ shows this to be so. For instance, define $<$ by $x < y$, $z < w$, $x < w$. It is easily seen that $(X,\sim)$ is both an interval graph and a comparability graph and that the strong property preceding Theorem 7.3 does not hold for $<$.

9. SUMMARY OF THE TESTS FOR TWO DIMENSIONALITY

In this section, we compile all the tests for two dimensionality of a partial order $(X,<)$ which were mentioned earlier or given by Dushnik and Miller [1941]. It is sufficient to summarize these tests for the case where X is finite, since $D(<) \leq 2$ if and only if $D(<') \leq 2$ for all finite restrictions $<'$ of $<$.

Before listing the tests, we introduce one further definition. If $(X,<*)$ is a linear order extending the partial order $(X,<)$, and $\sim$ is the incomparability relation for $<$, we say $<*$ is a <u>nonseparating extension</u> if $x <* y <* z$ and $x \sim y \sim z$ implies $x \sim z$. By a theorem of Dushnik and Miller, $D(<) \leq 2$ if and only if $<$ has a nonseparating linear extension. For example, if $(X,<)$ is the partial order "below and to the left" on the plane $X = R^2$, then a nonseparating linear extension is the order "below" on R^2. (In general, a nonseparating linear extension of a partial order of $D \leq 2$ is obtained by taking the union of the partial order and its conjugate.) Now summarizing, we have:

THEOREM 9.1. *Suppose $<$ is a partial order on a finite set X and $\sim$ is its incomparability relation. Then all of the following statements are equivalent:*

(a) $D(<) \leq 2$.

(b) There is a conjugate partial order $<*$ on X.

(c) There is a nonseparating linear extension of $<$.

(d) $(X,\sim)$ is a comparability graph.

(e) Every odd $\sim$ -cycle has a triangular chord.

(f) $(X,<)$ has no comparability cycle.

(g) $L(X,<)$ has a planar Hasse diagram.

(h) $<$ is realizable as the partial order of inclusion on a set of intervals in some linear order.

(i) $<$ satisfies the weak interval condition.

10. BREADTH OF A PARTIAL ORDER

We shall conclude this study with a few remarks on the breadth of a partial order, a concept that stems from Dilworth's work [1940] in lattice theory and one that is related to dimensionality. Additional theory on breadth is presented by Baker [1961].

Given a partially ordered set $(X,<)$, we write $\leqq$ for "$x < y$ or $x = y$." A subset A of X is said to be <u>join-independent</u> (or union independent) if and only if $a \in A$ implies that there is an $x \in X$ such that $a \nleqq x$ and $b \leqq x$ for all $b \in A - \{a\}$. In Fig. 3, $\{f\}$ is join-independent since $f \nleqq c$. Two-element join-independent subsets of Fig. 3 include all subsets $\{x,y\}$ with $x \sim y$. A necessary (but not sufficient) condition for $\{x_1,x_2,\ldots,x_n\}$ to be join-independent is that $x_i \sim x_j$ for all $i \neq j$. There is no three-element join-independent subset in the partial order of Fig. 3.

For a partially ordered set $(X,<)$, the <u>breadth</u> $B(<)$ is the supremum (least upper bound) of all cardinal numbers m such that X contains a join-independent subset of power m. Thus, like $D(<)$, $B(<)$ is defined for every partial order.

It follows from the two preceding paragraphs that $B(<) = 2$ for the partial order $<$ of Fig. 3, whereas $D(<) = 3$ in that case. Baker [1961] proves that it is always true that $B(<) \leq D(<)$ for a partial order $<$. It follows directly

from this and Theorem 7.3 that $B(<) \leq 2$ when $<$ is a strong interval order. Moreover, in contrast to the conclusion of Sec. 7 that $D(<)$ may exceed 2 for an interval order, we now prove

<u>THEOREM 10.1</u>. <u>If $<$ is an interval order then</u> $B(<) \leq 2$.

<u>Proof</u>. Suppose that $B(<) > 2$. Then X contains a join-independent subset Y with more than two elements. By the definition of join-independence, any three-element subset of Y will constitute the lower half of a crown of order three in X. (See $<_3$ of Fig. 2.) But a crown of order three is clearly not an interval order, hence cannot be the restriction of an interval order. Thus $<$ is not an interval order.

It is easily seen from $\{x < y,\ z < w,\ x \sim w,\ y \sim z\}$ that when $B(<) \leq 2$, $<$ need not be an interval order.

In contrast to the theorems of Sec. 5 on the axiomatization of $D \leq 2$, the proof just concluded shows that the condition $B \leq 2$ does have a finite axiomatization: $B(<) \leq 2$ if and only if no restriction of $<$ is a crown of order 3. More generally, for any cardinal m, $B(<) \leq m$ if and only if no restriction of $<$ is isomorphic to the partial order of inclusion on the set of singleton subsets and their complements in a set with n elements, where n is the cardinal following m. ($n = m+1$ if M is finite.)

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